EXOTIC DIFFERENTIAL STRUCTURES IN DIMENSION 2

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ABSTRACT. It is known that the long line supports $2^{\aleph_1}$ many non-diffeomorphic differential structures. We show that the long plane supports a similar number of exotic differential structures, ie structures which are not merely diffeomorphic to the product of two structures on the factor spaces.

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1. INTRODUCTION

In this paper, by a *differential structure* we mean a C^r differential structure for any $r \geq 1$. Recall, [3], that every C^r structure contains a C^s structure for any s satisfying $r < s \leq \infty$. Hence we are not concerned about the value of r.

It is well-known that euclidean space, $\mathbb{R}^n$, possesses a unique differential structure up to diffeomorphism for $n \neq 4$ but $\mathbb{R}^4$ has $\mathfrak{c}$ mutually non-diffeomorphic differential structures; see [2, page 95] for early details and [5] for a recent survey. Thus $\mathbb{R}^4$ possesses many *exotic differential structures*, i.e., differential structures which are not diffeomorphic to the 4-fold product of $\mathbb{R}$ with the usual structure (or the 2-fold product of $\mathbb{R}^2$ with the usual structure). Of course exotic differential structures were discovered more than half a century ago by Milnor in [4] where there is given the first construction of a differential structure on the 7-sphere $\mathbb{S}^7$ which is not diffeomorphic to the usual (product) differential structure inherited from $\mathbb{R}^8$. The existence of two mutually non-diffeomorphic differential structures on a manifold is not possible for metrisable manifolds in dimension up to 3, [5]: this result is due to Radó in dimensions 1 and 2 and Moise in dimension 3.

On the other hand in [8] it is shown that when we relax the metrisability condition then even in dimension 1 there are $2^{\aleph_1}$ mutually non-diffeomorphic differential structures (on the long ray, hence also the long line $\mathbb{L}$). As a result there are also $2^{\aleph_1}$ mutually non-diffeomorphic differential structures on the long plane $\mathbb{L}^2$. In this paper we address the question: does the long plane support differential structures which are not diffeomorphic to any product structure? Our answer is "yes."

As usual we denote by ω_1 the set of countable ordinals with the order topology. Let $\mathbb{L}_{\geq 0}$ denote the *closed long ray*, ie the set $\omega_1 \times [0, 1)$ with the lexicographic order topology, and let $\mathbb{L}$ denote the *long line* which is obtained from two copies of the closed long ray with their initial points identified to 0. The *(open) long ray* is the 1-manifold $\mathbb{L}_+ = \mathbb{L}_{\geq 0} - \{(0, 0)\}$. Identify $\alpha \in \omega_1$ with $(\alpha, 0) \in \mathbb{L}_{\geq 0}$. We will exhibit non-product differential structures on $\mathbb{L}^2_+$. As in [8] similar structures may then be deduced on $\mathbb{L}^2$.

The following result is well-known and is found in many books introducing Set Theory but we include it for completeness. Note that it does not matter whether we are considering C and D as subsets of ω_1 or $\mathbb{L}_+$.

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Lemma 1. *If $C, D \subset \omega_1$ are closed unbounded subsets then $C \cap D$ is also closed and unbounded.*

Proposition 2. *Suppose that $\mathcal{D}$ is a differential structure on $\mathbb{L}_+$ and $\alpha \in \mathbb{L}_+$. Then $((\alpha, \omega_1), \mathcal{D}|(\alpha, \omega_1))$ is diffeomorphic to $(\mathbb{L}_+, \mathcal{D})$.*

Proof. Choose $\beta, \gamma \in \mathbb{L}_+$ such that $\alpha < \beta < \gamma$. Because $\mathbb{R}$ has a unique differential structure up to diffeomorphism and $(0, \gamma) \subset \mathbb{L}_+$ is homeomorphic to $\mathbb{R}$ we may choose a diffeomorphism $g : ((0, \gamma), \mathcal{D}|(0, \gamma)) \to ((0, 3), \mathcal{U})$, where $\mathcal{U}$ is the usual differential structure on $\mathbb{R}$ restricted to $(0, 3)$. Furthermore we may assume that $g(\alpha) = 1$ and $g(\beta) = 2$. Next let $\theta : (0, 3) \to (1, 3)$ be a diffeomorphism (relative to $\mathcal{U}$) such that $\theta(t) = t$ for all $t \in [2, 3)$. For example set $\theta(t) = \begin{cases} t + \sqrt{e}e^{\frac{1}{t-2}} & \text{if} \quad t < 2 \\ t & \text{if} \quad t \geq 2 \end{cases}$. Now define $h : \mathbb{L}_+ \to (\alpha, \omega_1)$ by $h(t) = \begin{cases} g^{-1}\theta g(t) & \text{if} \quad t < \gamma \\ t & \text{if} \quad t > \beta \end{cases}$. Then h is a diffeomorphism with respect to the structure $\mathcal{D}$. $\square$

Recall the following result from [1, Corollary 2.6].

Proposition 3. *Suppose that $h : \mathbb{L}_+^2 \to \mathbb{L}_+^2$ is an orientation-preserving homeomorphism. Then $\{\alpha \in \omega_1 \ / \ h(\mathbb{L}_+ \times \{\alpha\}) = \mathbb{L}_+ \times \{\alpha\}\}$ is a closed unbounded set.*

Corollary 4. *Suppose that $\mathcal{F}$ is a differential structure on $\mathbb{L}_+^2$ and that $\mathcal{F}$ is diffeomorphic to the product of two structures. Then*

$$\{\alpha \in \omega_1 \ / \ \mathbb{L}_+ \times \{\alpha\} \textit{ is a differentiable submanifold of } (\mathbb{L}_+^2, \mathcal{F})\}$$

and

$$\{\alpha \in \omega_1 \ / \ \{\alpha\} \times \mathbb{L}_+ \textit{ is a differentiable submanifold of } (\mathbb{L}_+^2, \mathcal{F})\}$$

are closed unbounded subsets of ω_1.

Proof. There are two differential structures, say $\mathcal{D}, \mathcal{E}$, on $\mathbb{L}_+$ and a diffeomorphism $h : (\mathbb{L}_+, \mathcal{D}) \times (\mathbb{L}_+, \mathcal{E}) \to (\mathbb{L}_+^2, \mathcal{F})$. Interchanging the roles of $\mathcal{D}$ and $\mathcal{E}$ if necessary we may assume h preserves orientation. By Proposition 3, $S = \{\alpha \in \omega_1 \ / \ h(\mathbb{L}_+ \times \{\alpha\}) = \mathbb{L}_+ \times \{\alpha\}\}$ is a closed unbounded set. Thus $\mathbb{L}_+ \times \{\alpha\}$ is a differentiable submanifold of $(\mathbb{L}_+^2, \mathcal{F})$ for each $\alpha \in S$. Interchanging the coordinates leads to the other half. $\square$

We also require the following folklore result, cf [7, Theorem 1] and [6, Theorem 3 page 46].

Proposition 5. *Let $M \subset \mathbb{R}^2$ be a compact topological manifold with boundary, $K \subset M$ a compact subset which contains the boundary of M and suppose that $h : M \to M$ is a homeomorphism which is a diffeomorphism on a neighbourhood of K. Then h can be approximated arbitrarily closely by a homeomorphism which is a diffeomorphism on $\mathring{M}$ and agrees with h on a neighbourhood of K.*

2. Exotic Differential Structures on $\mathbb{L}_+^2$

We now present a method of constructing from two differential structures on the long ray a differential structure on $\mathbb{L}_+^2$ which is not diffeomorphic to the product of any two differential structures on $\mathbb{L}_+$. The construction allows us to verify that there are $2^{\aleph_1}$ many non-diffeomorphic such structures.

We require an auxiliary shearing homeomorphism $\sigma : [0, 5]^2 \to [0, 5]^2$. The homeomorphism σ is the identity except in the rectangle $(3, 4) \times (1, 4)$, does not change the first coordinate and maps the straight line segment $[3, 4] \times \{2\}$ onto the two line segments $\left\{\left(x, 3 - 2\left|x - 3\frac{1}{2}\right|\right) \ : \ 3 \leq x \leq 4\right\}$. The notation $I_\alpha = (0, \alpha + 1)$, $\overline{I_\alpha} = [0, \alpha + 1]$, $O_\alpha = I_\alpha^2 \subset \mathbb{L}_+^2$ and $\overline{O_\alpha} = \overline{I_\alpha}^2 \subset \mathbb{L}_{\geq 0}^2$ is fixed throughout this section.

Begin with two differential structures $\mathcal{D}$ and $\mathcal{E}$ on $\mathbb{L}_+$; for example any of those in [8] will do. For each $\alpha \in \omega_1 \setminus \{0\}$ choose order-preserving homeomorphisms $\psi_\alpha, \chi_\alpha : \overline{I_\alpha} \to [0,5]$ so that $\psi_\alpha(\alpha) = \chi_\alpha(\alpha) = 2$, and that $(I_\alpha, \psi_\alpha) \in \mathcal{D}$ and $(I_\alpha, \chi_\alpha) \in \mathcal{E}$.

For each $\alpha \in \omega_1 \setminus \{0\}$ we will construct by induction on α a homeomorphism $\varphi_\alpha : \overline{O_\alpha} \to [0,5]^2$ in such a way that $\{(O_\alpha, \varphi_\alpha) \ / \ \alpha \in \omega_1 \setminus \{0\}\}$ is a basis for a differential structure on $\mathbb{L}_+^2$, i.e., for each $\alpha, \beta \in \omega_1 \setminus \{0\}$ the maps $\varphi_\alpha \varphi_\beta^{-1}$ and $\varphi_\beta \varphi_\alpha^{-1}$ are smooth where defined within $(0,5)^2$. The induction includes the further condition:

- The homeomorphisms $\psi_\alpha \times \chi_\alpha$ and φ_α agree on neighbourhoods of $\overline{O_\alpha} \setminus O_\alpha$ and of $[0,\alpha) \times [\alpha, \alpha+1]$ as well as on a neighbourhood in $\overline{O_\alpha} \setminus [0,\alpha)^2$ of $\{\alpha\} \times [0,\alpha]$.

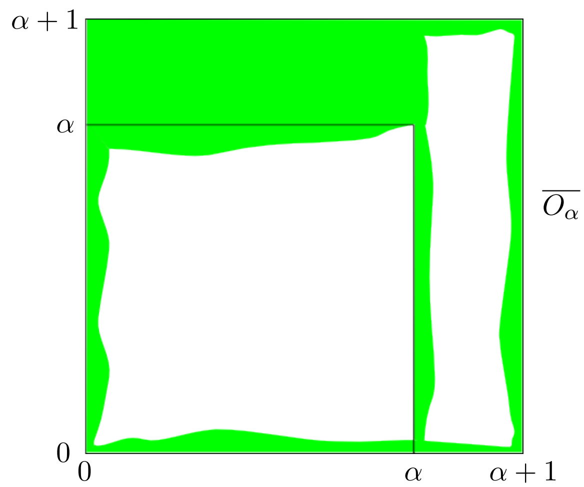

FIGURE 1. Where $\varphi_\alpha = \psi_\alpha \times \chi_\alpha$.

Definition of φ_1:- Set $\varphi_1 = \sigma(\psi_1 \times \chi_1)$.
Definition of $\varphi_{\alpha+1}$ given φ_α:- Define

$$\varphi_{\alpha+1}(z) = \begin{cases} (\psi_{\alpha+1} \times \chi_{\alpha+1})(\psi_\alpha \times \chi_\alpha)^{-1}\varphi_\alpha(z) & \text{if} \quad z \in \overline{O_\alpha}; \\ \sigma(\psi_{\alpha+1} \times \chi_{\alpha+1})(z) & \text{if} \quad z \in \overline{O_{\alpha+1}} - O_\alpha \end{cases} .$$

It is easily checked that the inductive conditions are satisfied.
Definition of φ_α, where α is a limit ordinal, given φ_β for all $\beta \in \omega_1 \setminus \{0\}$ with $\beta < \alpha$:- Firstly choose some metric d on $\overline{O_\alpha}$ compatible with the topology. Next choose an increasing sequence $\langle \alpha_n \rangle$ from $\omega_1 \setminus \{0\}$ converging to α; set $\alpha_0 = 0$. Somewhat as in the previous case we would like to let φ_α be $(\psi_\alpha \times \chi_\alpha)(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}\varphi_{\alpha_n}$ on $\overline{O_{\alpha_n}}$ and be $\sigma(\psi_\alpha \times \chi_\alpha)$ outside the union of these common domains but this would work only if all maps of the form $(\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}\varphi_{\alpha_m}$ and $(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}\varphi_{\alpha_n}$ agree on $\overline{O_{\alpha_{\min\{m,n\}}}}$. We modify these maps inductively so that they do agree, at least on enough of $\overline{O_{\alpha_{\min\{m,n\}}}}$. To effect this we construct a sequence of homeomorphisms $\langle h_n : [0,3]^2 \to [0,3]^2 \rangle$, where $n \geq 1$. We demand the following properties:

- $h_n : (0,3)^2 \to (0,3)^2$ is a diffeomorphism;
- h_n is the identity on a neighbourhood of $([0,3] \times [2,3]) \cup ([2,3] \times [0,3])$;
- $h_n = (\psi_{\alpha_n} \times \chi_{\alpha_n})(\psi_{\alpha_{n-1}} \times \chi_{\alpha_{n-1}})^{-1} h_{n-1} \varphi_{\alpha_{n-1}} \varphi_{\alpha_n}^{-1}$ on a neighbourhood of $\varphi_{\alpha_n}([0,\alpha_{n-1})^2)$ when $n > 1$;
- $h_n = (\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi_{\alpha_n}^{-1}$ on a neighbourhood of $\varphi_{\alpha_n}([0,\alpha_{n-1}) \times [\alpha_{n-1}, \alpha_n])$ when $n > 1$;

- for $n > 1$, on $\varphi_{\alpha_n}([\alpha_{n-1}, \alpha_n] \times [0, \alpha_{n-2}])$, h_n is sufficiently close to $(\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi_{\alpha_n}^{-1}$ that for any $(x, y) \in \varphi_{\alpha_n}([\alpha_{n-1}, \alpha_n] \times [0, \alpha_{n-2}])$, we have

$$d\left((\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}(x, y), (\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n(x, y)\right) < \frac{1}{n}.$$

  To achieve this we use uniform continuity of $(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}$.

Notice that the conditions on h_n are mutually consistent by the inductively assumed conditions on φ_{α_m} and h_{n-1}.

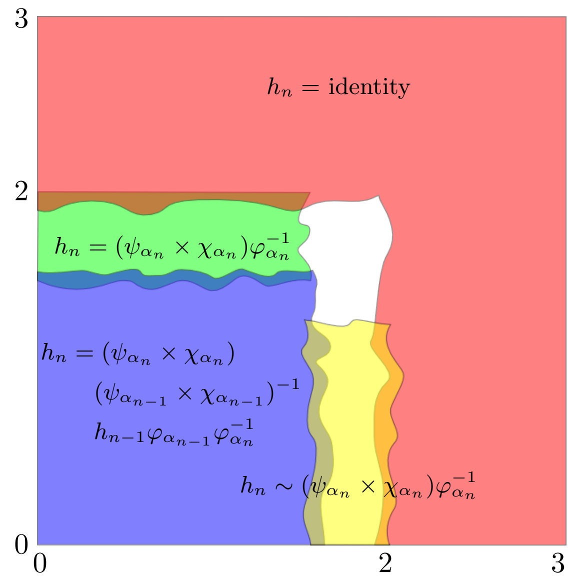

FIGURE 2. Constraints on h_n.

Let $h_1 : [0, 3]^2 \to [0, 3]^2$ be the identity. Suppose given $n > 1$ such that h_{n-1} has been defined. By Proposition 5 there is a homeomorphism $h_n : [0, 3]^2 \to [0, 3]^2$ satisfying the requirements.

Now define φ_α by

$$\varphi_\alpha(x) = \begin{cases} (\psi_\alpha \times \chi_\alpha)(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}(x) & \text{if} \quad x \in [0, \alpha_n]^2 \text{ for some } n \\ \sigma(\psi_\alpha \times \chi_\alpha)(x) & \text{if} \quad x \in \overline{O_\alpha} \setminus (0, \alpha)^2 \end{cases}.$$

The function φ_α is well-defined because if $m < n$ then $(\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}h_m\varphi_{\alpha_m}$ and $(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}$ agree on $[0, \alpha_m]^2$. It is easily verified that φ_α is a homeomorphism, the main problem being to verify continuity on $\{\alpha\} \times [0, \alpha]$. It is here that we require precision in the approximation of the homeomorphism $(\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi_{\alpha_n}^{-1}$ by the diffeomorphism as required in the last inductive assumption for h_n. The approximation must improve as n increases so that any sequence $\langle(x_n, y_n)\rangle$ in $[0, \alpha)^2$ converging to $(\alpha, y) \in \{\alpha\} \times [0, \alpha]$ is mapped by $(\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}h_m\varphi_{\alpha_m}$ (m increasing with n) to a sequence which also converges to (α, y). Then $\varphi_\alpha(x_n, y_n) \to (\psi_\alpha \times \chi_\alpha)(\alpha, y) = \varphi(\alpha, y)$ as σ is the identity on $\{2\} \times [0, 5]$.

Suppose $\beta < \alpha$. Then the coordinate transformation between φ_α and φ_β is smooth on $\varphi_\beta(O_{\beta+1})$ because

$$\varphi_\alpha\varphi_\beta^{-1} = (\psi_\alpha \times \chi_\alpha)(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}\varphi_\beta^{-1}$$

is a composition of coordinate transition functions together with the diffeomorphism h_n and hence is smooth, where n is chosen so that $\alpha_n > \beta$. Similarly its inverse is smooth.

The remaining condition demanded of φ_α is also satisfied.

Thus we have constructed a basis $\{(O_\alpha, \varphi_\alpha) \ / \ \alpha \in \omega_1 \setminus \{0\}\}$ for a differential structure on $\mathbb{L}_+^2$. Call this structure $\mathcal{F}$.

Claim 6. *The differential structure $\mathcal{F}$ is not diffeomorphic to a product of structures on $\mathbb{L}_+$.*

Proof. Let $\alpha < \omega_1$ be any non-zero ordinal. We first show that $\mathbb{L}_+ \times \{\alpha\}$ is not a smooth submanifold of $(\mathbb{L}_+^2, \mathcal{F})$. Suppose instead that $\mathbb{L}_+ \times \{\alpha\}$ is a smooth submanifold of $(\mathbb{L}_+^2, \mathcal{F})$. Then $(\alpha, \alpha+1) \times \{\alpha\}$ is also a smooth submanifold of $(\mathbb{L}_+^2, \mathcal{F})$, so there is a chart $\Big((\alpha, \alpha+1) \times (0, \alpha+1), \varphi\Big) \in \mathcal{F}$ such that $\varphi^{-1}(\mathbb{R} \times \{0\}) = (\alpha, \alpha+1) \times \{\alpha\}$. It follows that $\varphi_\alpha \varphi^{-1}(\mathbb{R} \times \{0\}) = \varphi_\alpha((\alpha, \alpha+1) \times \{\alpha\})$ is a smooth submanifold of $\mathbb{R}^2$ with the usual differential structure. However, for $t \in (\alpha, \alpha+1)$ we have $\varphi_\alpha(t, \alpha) = \sigma(\psi_\alpha(t), 2)$, and hence

$$\varphi_\alpha((\alpha, \alpha+1) \times \{\alpha\}) = ((2,3] \cup [4,5)) \times \{2\} \cup \left\{ \left(x, 3 - 2\left| x - 3\frac{1}{2} \right| \right) \ : \ 3 \le x \le 4 \right\}.$$

As this last set is not a smooth submanifold of $\mathbb{R}^2$, it follows that $(0, \omega_1) \times \{\alpha\}$ is not a smooth submanifold of $(\mathbb{L}_+^2, \mathcal{F})$.

The claim now follows from Lemma 1 and Corollary 4 because $\omega_1 \setminus \{0\}$ is closed and unbounded. ∎

We now address the question: how many exotic differential structures does $\mathbb{L}_+^2$ support? The argument presented in [8, p.156] that $\mathbb{L}_+$ supports no more than $2^{\aleph_1}$ many mutually non-diffeomorphic differential structures applies as well to $\mathbb{L}_+^2$. On the other hand [8, Theorem 5.2] exhibits exactly $2^{\aleph_1}$ many mutually non-diffeomorphic differential structures on $\mathbb{L}_+$. Thus we might expect to find $2^{\aleph_1}$ many mutually non-diffeomorphic exotic differential structures on $\mathbb{L}_+^2$, and this is indeed the case.

Let $\mathcal{D}$ be any differential structure on $\mathbb{L}_+$. Apply the construction above with $\mathcal{E} = \mathcal{D}$ and denote the resulting exotic differential structure on $\mathbb{L}_+^2$ by $\widehat{\mathcal{D}}$.

Theorem 7. *There are $2^{\aleph_1}$ mutually non-diffeomorphic exotic differential structures on $\mathbb{L}_+^2$.*

Proof. Suppose given differential structures $\mathcal{D}$ and $\mathcal{E}$ on $\mathbb{L}_+$ and an orientation-preserving diffeomorphism $h : (\mathbb{L}_+^2, \widehat{\mathcal{D}}) \to (\mathbb{L}_+^2, \widehat{\mathcal{E}})$. By Proposition 3, for a closed unbounded set of $\alpha \in \omega_1$, the map h restricts to a homeomorphism taking $\{\alpha\} \times (\alpha, \omega_1)$ to itself. Now $\widehat{\mathcal{D}} | \{\alpha\} \times (\alpha, \omega_1)$ is the same as $\mathcal{D} \times \mathcal{D} | \{\alpha\} \times (\alpha, \omega_1)$ with the same for $\mathcal{E}$ so, using Proposition 2 and denoting "is diffeomorphic to" by $\approx$, we have

$$\begin{aligned}
(\mathbb{L}_+, \mathcal{D}) &\approx ((\alpha, \omega_1), \mathcal{D}) \\
&\approx (\{\alpha\} \times (\alpha, \omega_1), \mathcal{D} \times \mathcal{D} | \{\alpha\} \times (\alpha, \omega_1)) \\
&\approx (\{\alpha\} \times (\alpha, \omega_1), \widehat{\mathcal{D}} | \{\alpha\} \times (\alpha, \omega_1)) \\
&\approx (\{\alpha\} \times (\alpha, \omega_1), \widehat{\mathcal{E}} | \{\alpha\} \times (\alpha, \omega_1)) \\
&\approx (\{\alpha\} \times (\alpha, \omega_1), \mathcal{E} \times \mathcal{E} | \{\alpha\} \times (\alpha, \omega_1)) \\
&\approx ((\alpha, \omega_1), \mathcal{E}) \\
&\approx (\mathbb{L}_+, \mathcal{E}).
\end{aligned}$$

It follows that for any differential structure $\mathcal{D}$ on $\mathbb{L}_+$ there can be at most one equivalence class of structures, represented say by $\mathcal{E}$, such that $(\mathbb{L}_+^2, \widehat{\mathcal{D}})$ is diffeomorphic to $(\mathbb{L}_+^2, \widehat{\mathcal{E}})$ but $(\mathbb{L}_+, \mathcal{D})$ is not diffeomorphic to $(\mathbb{L}_+, \mathcal{E})$. Indeed, if $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are three differential structures on $\mathbb{L}_+$ and $(\mathbb{L}_+^2, \widehat{\mathcal{D}})$ is diffeomorphic to both $(\mathbb{L}_+^2, \widehat{\mathcal{E}})$ and $(\mathbb{L}_+^2, \widehat{\mathcal{F}})$ but $(\mathbb{L}_+, \mathcal{D})$ is not diffeomorphic to either $(\mathbb{L}_+, \mathcal{E})$ or $(\mathbb{L}_+, \mathcal{F})$, then diffeomorphisms $g : (\mathbb{L}_+^2, \widehat{\mathcal{D}}) \to (\mathbb{L}_+^2, \widehat{\mathcal{E}})$ and $h : (\mathbb{L}_+^2, \widehat{\mathcal{D}}) \to (\mathbb{L}_+^2, \widehat{\mathcal{F}})$ must reverse

orientation. In that case the diffeomorphism $hg^{-1} : (\mathbb{L}_+^2, \widehat{\mathcal{E}}) \to (\mathbb{L}_+^2, \widehat{\mathcal{F}})$ preserves orientation and hence $(\mathbb{L}_+, \mathcal{E})$ is diffeomorphic to $(\mathbb{L}_+, \mathcal{F})$ by what we have already shown.

It now follows from [8, Theorem 5.2] that there are $2^{\aleph_1}$ mutually non-diffeomorphic exotic differential structures on $\mathbb{L}_+^2$. $\square$

As a complement to Theorem 7 we have the following.

Theorem 8. *There are $2^{\aleph_1}$ mutually non-diffeomorphic product differential structures on $\mathbb{L}_+^2$.*

Proof. It suffices to show that if $\mathcal{D}$ and $\mathcal{E}$ are two differential structures on $\mathbb{L}$ such that $(\mathbb{L}^2, \mathcal{D} \times \mathcal{D})$ is diffeomorphic to $(\mathbb{L}^2, \mathcal{E} \times \mathcal{E})$ then $(\mathbb{L}, \mathcal{D})$ is diffeomorphic to $(\mathbb{L}, \mathcal{E})$. However it is easy to show that the homeomorphism $(x, x) \mapsto x$ from the diagonal Δ of $\mathbb{L}^2$ to $\mathbb{L}$ is a diffeomorphism from $(\Delta, \mathcal{D} \times \mathcal{D}|\Delta)$ to $(\mathbb{L}, \mathcal{D})$. $\square$

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